
# Recursion

# Outline

- Induction
- Linear recursion
  - Example 1: Factorials
  - Example 2: Powers
  - Example 3: Reversing an array
- Binary recursion
  - Example 1: The Fibonacci sequence
  - Example 2: The Tower of Hanoi
- Drawbacks and pitfalls of recursion

# Outcomes

- By understanding this lecture you should be able to:
  - Use induction to prove the correctness of a recursive algorithm.
  - Identify the base case for an inductive solution
  - Design and analyze linear and binary recursion algorithms
  - Identify the overhead costs of recursion
  - Avoid errors commonly made in writing recursive algorithms

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# Divide and Conquer

- When faced with a difficult problem, a classic technique is to break it down into smaller parts that can be solved more easily.
- Recursion uses induction to do this.

# History of Induction

- Implicit use of induction goes back at least to Euclid's proof that the number of primes is infinite (c. 300 BC).
- The first explicit formulation of the principle is due to Pascal (1665).

Euclid of Alexandria,
"The Father of Geometry"
c. 300 BC

Blaise Pascal, 1623 - 1662

# Induction: Review

- Induction is a mathematical method for proving that a statement is true for a (possibly infinite) sequence of objects.
- There are two things that must be proved:
  1. **The Base Case**: The statement is true for the first object
  2. **The Inductive Step**: If the statement is true for a given object, it is also true for the next object.
- If these two statements hold, then the statement holds for all objects.

# Induction Example: An Arithmetic Sum

- Claim: $\sum_{i=0}^{n} i = \frac{1}{2}n(n+1) \quad \forall n \in \mathbb{N}$
- Proof:

1. **Base Case.** The statement holds for $n = 0$:

$$\sum_{i=0}^{n} i = \sum_{i=0}^{0} i = 0$$

$$\frac{1}{2}n(n+1) = \frac{1}{2}0(0+1) = 0 \qquad ✓$$

2. **Inductive Step.** If the claim holds for $n = k$, then it also holds for $n = k+1$.

$$\sum_{i=0}^{k+1} i = k+1+\sum_{i=0}^{k} i = k+1+\frac{1}{2}k(k+1) = \frac{1}{2}(k+1)(k+2) \qquad ✓$$

# Recursive Divide and Conquer

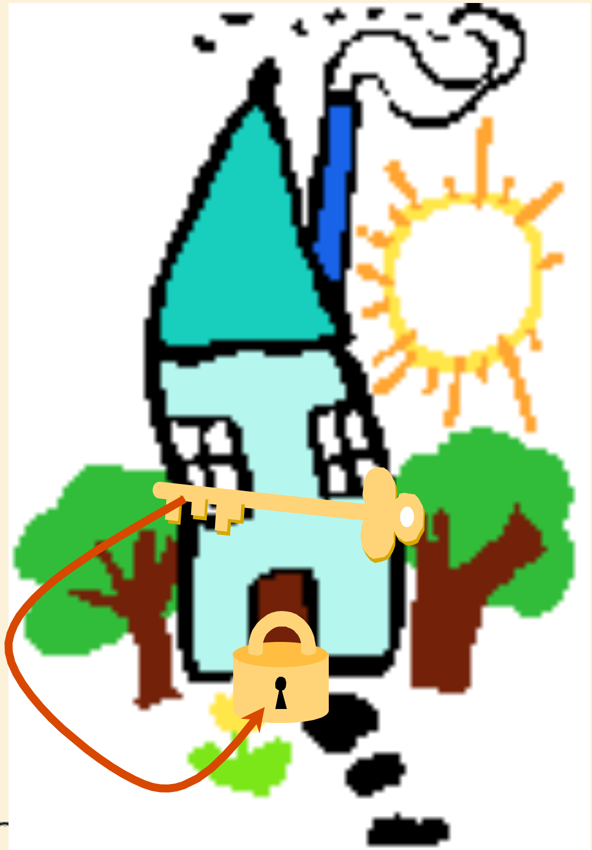
- You are given a problem input that is too big to solve directly.
- You imagine,
  - "Suppose I had a friend who could give me the answer to the same problem with slightly smaller input."
  - "Then I could easily solve the larger problem."
- In recursion this "friend" will actually be another instance (clone) of yourself.

Tai (left) and Snuppy (right): the first puppy clone.

# Friends & Induction

Recursive Algorithm:

- Assume you have an algorithm that works.
- Use it to write an algorithm that works.



If I could get in,
I could get the key.
Then I could unlock the door
so that I can get in.

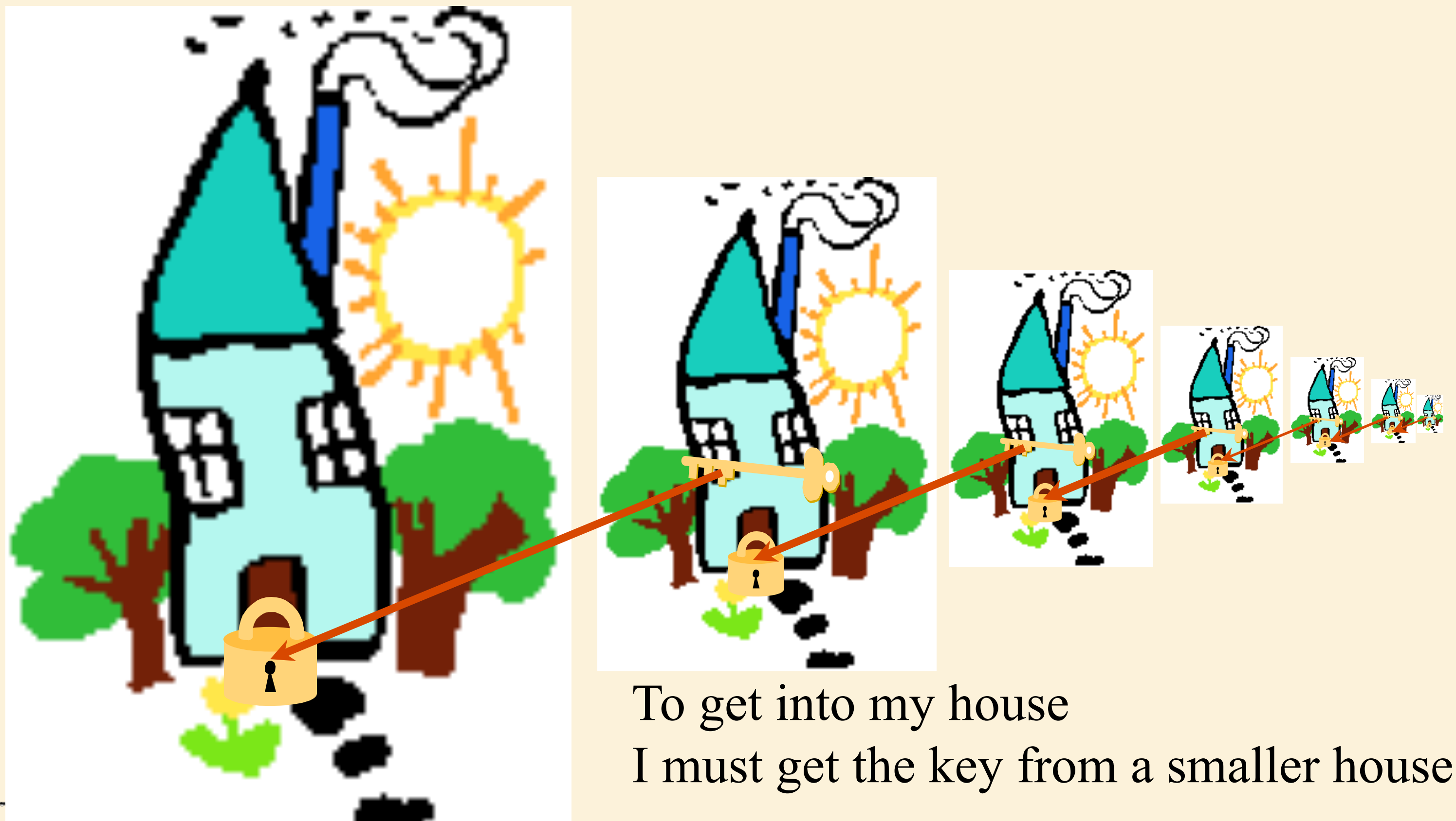
Circular Argument!

Example from J. Edmonds – Thanks Jeff!

# Friends & Induction

Recursive Algorithm:

- Assume you have an algorithm that works.
- Use it to write an algorithm that works.



To get into my house
I must get the key from a smaller house

# Friends & Induction

Recursive Algorithm:

- Assume you have an algorithm that works.
- Use it to write an algorithm that works.

The "base case"

Use brute force
to get into
the smallest house.

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# Recall: Design Pattern

- A template for a software solution that can be applied to a variety of situations.
- Main elements of solution are described in the abstract.
- Can be specialized to meet specific circumstances.

# Linear Recursion Design Pattern

- **Test for base cases**
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls **must** eventually reach a base case, and the handling of each base case should not use recursion.
- ***Recurse once***
  - Perform a single recursive call. (This recursive step may involve a test that decides which of several possible recursive calls to make, but it should ultimately choose to make just one of these calls each time we perform this step.)
  - Define each possible recursive call so that it makes **progress** towards a base case.

# Example 1

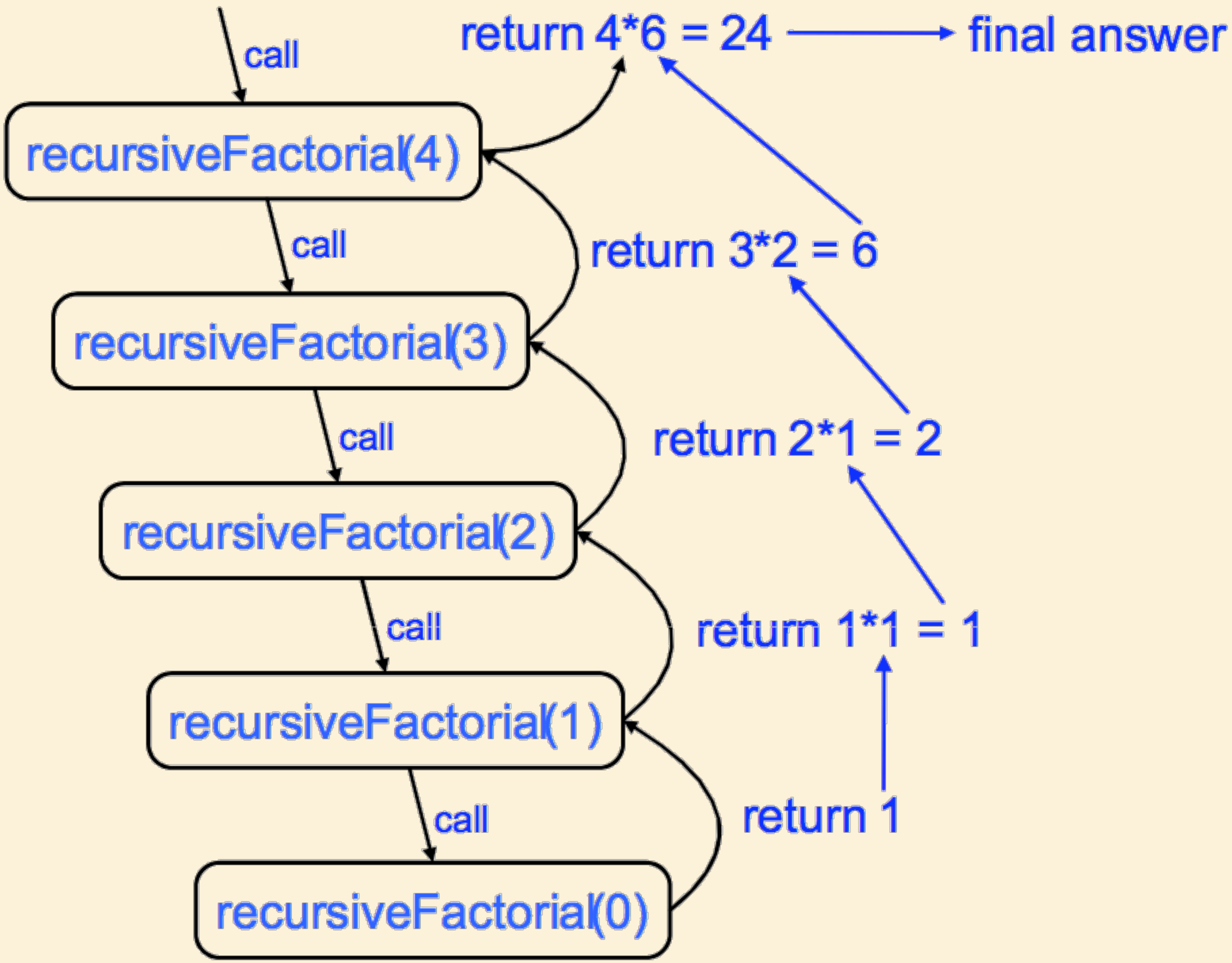
- The factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a Java method:

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;      // base case
    else return  n  *  recursiveFactorial(n- 1); // recursive case
}
```

# Tracing Recursion



# Linear Recursion

- recursiveFactorial is an example of **linear** recursion: only one recursive call is made per stack frame.
- Since there are *n* recursive calls, this algorithm has O(*n*) run time.

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
   if  (n  ==  0)  return  1;      // base case
   else return  n  *  recursiveFactorial(n- 1); // recursive case
}
```

# End of Lecture

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# Example 2: Computing Powers

- The power function, ***p(x,n) = x^n***, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{otherwise} \end{cases}$$

- Assume multiplication takes constant time (independent of value of arguments).
- This leads to a power function that runs in O(n) time (for we make n recursive calls).
- **Can we do better than this?**

# Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } n > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } n > 0 \text{ is even} \end{cases}$$

- For example,

$$2^4 = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$$

Naïve method entails 3 multiplies.

Recursive squaring entails 2 multiplies.

$$2^5 = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$$

Naïve method entails 4 multiplies.

Recursive squaring entails 3 multiplies.

# A Recursive Squaring Method

**Algorithm** Power($x$, $n$):

***Input:*** A number $x$ and integer $n$

***Output:*** The value $x^n$

**if** $n = 0$ **then**

**return** 1

**if** $n$ is odd **then**

$y = \text{Power}(x, (n - 1)/2)$

**return** $x \cdot y \cdot y$

**else**

$y = \text{Power}(x, n/2)$

**return** $y \cdot y$

# Analyzing the Recursive Squaring Method

**Algorithm** Power($x$, $n$):

***Input:*** A number $x$ and integer $n = 0$

***Output:*** The value $x^n$

**if** $n = 0$ **then**

  **return** 1

**if** $n$ is odd **then**

  $y = \text{Power}(x, (n - 1)/2)$

  **return** $x \cdot y \cdot y$

**else**

  $y = \text{Power}(x, n/2)$

  **return** $y \cdot y$

Although there are 2 statements that recursively call Power, only one is executed per stack frame.

Each time we make a recursive call we halve the value of n (roughly).

Thus we make a total of log n recursive calls. That is, this method runs in O(log n) time.

# Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its **last** step.
- Such a method can easily be converted to an iterative method (which saves on some resources).

# Example: Recursively Reversing an Array

**Algorithm** ReverseArray(*A*, *i*, *j*):

***Input:*** An array *A* and nonnegative integer indices *i* and *j*

***Output:*** The reversal of the elements in *A* starting at index *i* and ending at *j*

**if** *i* < *j* **then**

Swap *A*[*i*] and *A*[*j*]

ReverseArray(*A*, *i* + 1, *j* - 1)

**return**

# Example: Iteratively Reversing an Array

**Algorithm** IterativeReverseArray(*A*, *i*, *j* ):

***Input:*** An array *A* and nonnegative integer indices *i* and *j*

***Output:*** The reversal of the elements in *A* starting at index *i* and ending at *j*

**while** $i < j$ **do**

Swap $A[i]$ and $A[j]$

$i = i + 1$

$j = j - 1$

**return**

# Defining Arguments for Recursion

- Solving a problem recursively sometimes requires passing additional parameters.
- **ReverseArray** is a good example: although we might initially think of passing only the array ***A*** as a parameter at the top level, lower levels need to know where in the array they are operating.
- Thus the recursive interface is **ReverseArray(A, i, j)**.
- We then invoke the method at the highest level with the message **ReverseArray(A, 1, n)**.

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# Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example 1: **The Fibonacci Sequence**

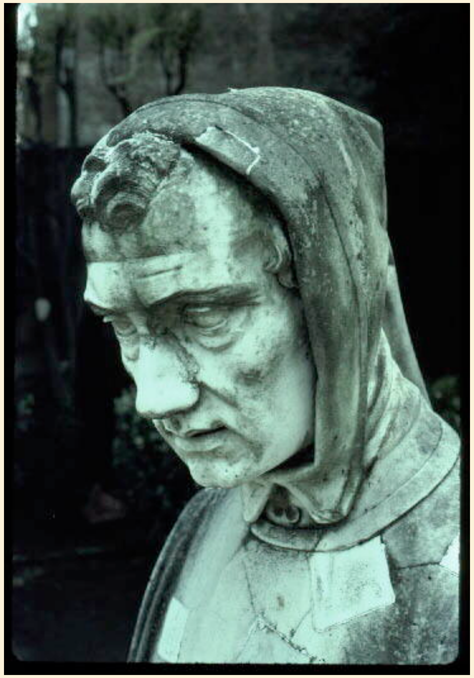
# The Fibonacci Sequence

- Fibonacci numbers are defined recursively:

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.



Fibonacci (c. 1170 - c. 1250)
(aka Leonardo of Pisa)

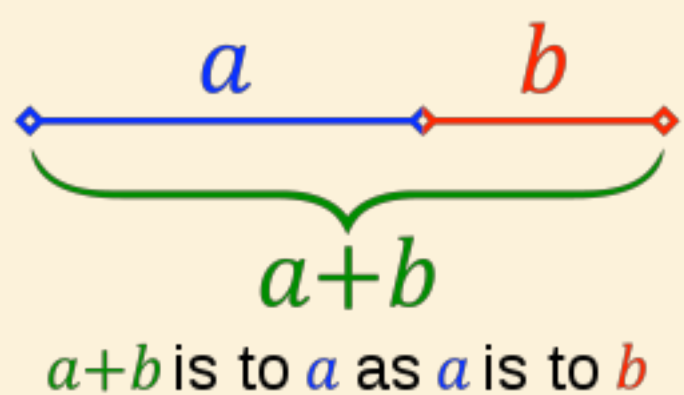
The ratio $F_i \,/\, F_{i-1}$ converges to $\varphi = \frac{1+\sqrt{5}}{2} = 1.61803398874989...$

(The "**Golden Ratio**")

# The Golden Ratio

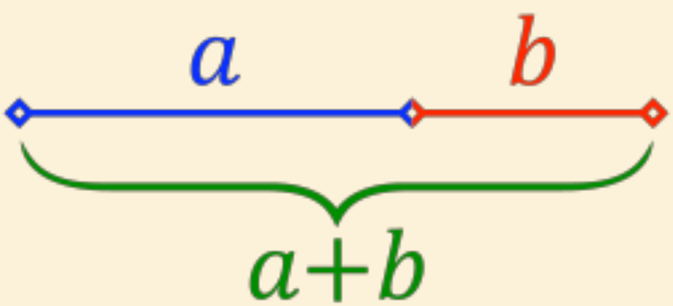
- Two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.

$\varphi$ is the unique positive solution to $\varphi = \dfrac{a+b}{a} = \dfrac{a}{b}$.

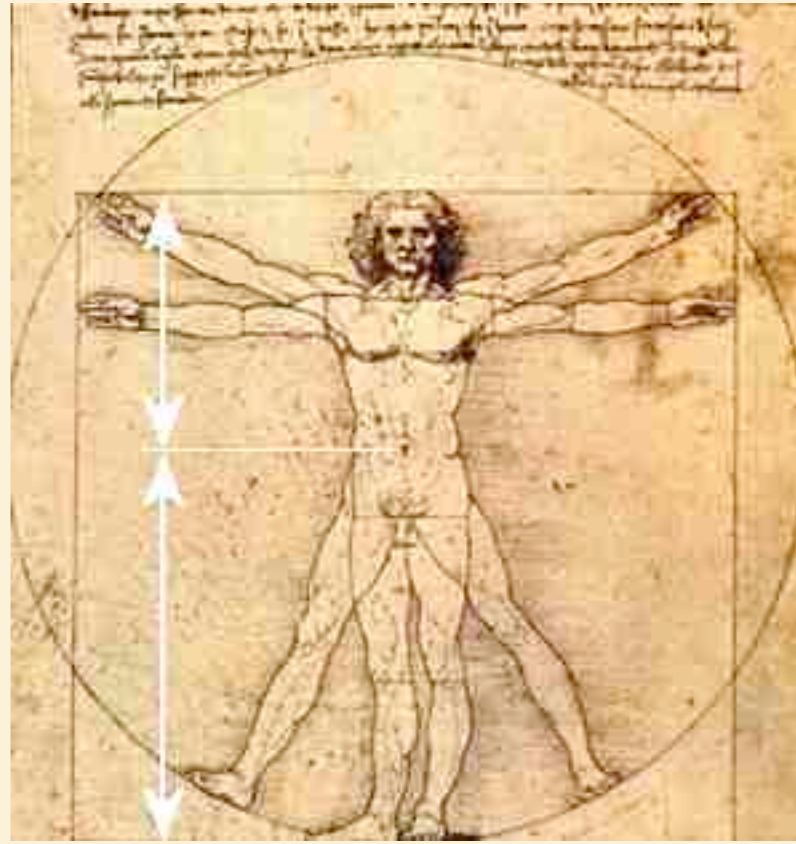


# The Golden Ratio

The Parthenon




$a+b$ is to $a$ as $a$ is to $b$



Leonardo

# Computing Fibonacci Numbers

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- A recursive algorithm (first attempt):

**Algorithm** BinaryFib($k$):

***Input:*** Positive integer $k$

***Output:*** The $k$th Fibonacci number $F_k$

**if** $k < 2$ **then**

**return** $k$

**else**

**return** BinaryFib($k$ - 1) + BinaryFib($k$ - 2)

# Analyzing the Binary Recursion Fibonacci Algorithm

- Let $n_k$ denote number of recursive calls made by BinaryFib(k). Then
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67.$
- Note that $n_k$ more than doubles for every other value of $n_k$. That is, $n_k > 2^{k/2}$. It increases exponentially!

# A Better Fibonacci Algorithm

- Use **linear** recursion instead:

**Algorithm** LinearFibonacci($k$):

***Input:*** A positive integer $k$

***Output:*** Pair of Fibonacci numbers $(F_k, F_{k-1})$

**if** $k = 1$ **then**

  **return** $(k, 0)$

**else**

  $(i, j)$ = LinearFibonacci($k$ - 1)

  **return** $(i + j, i)$

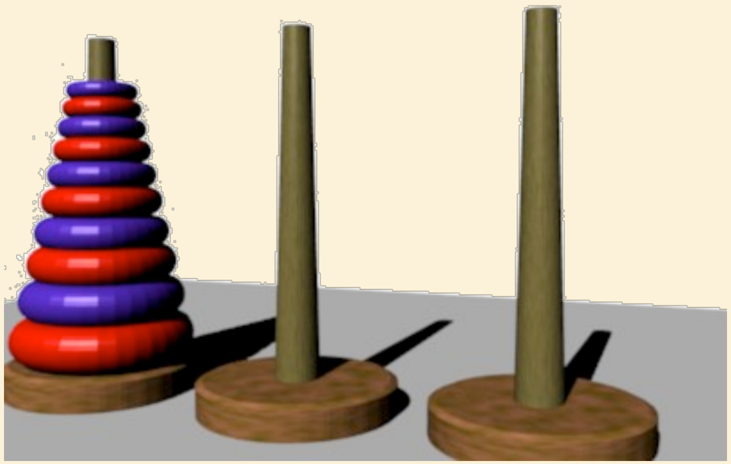
LinearFibonacci(k): $F_k$, $F_{k-1}$

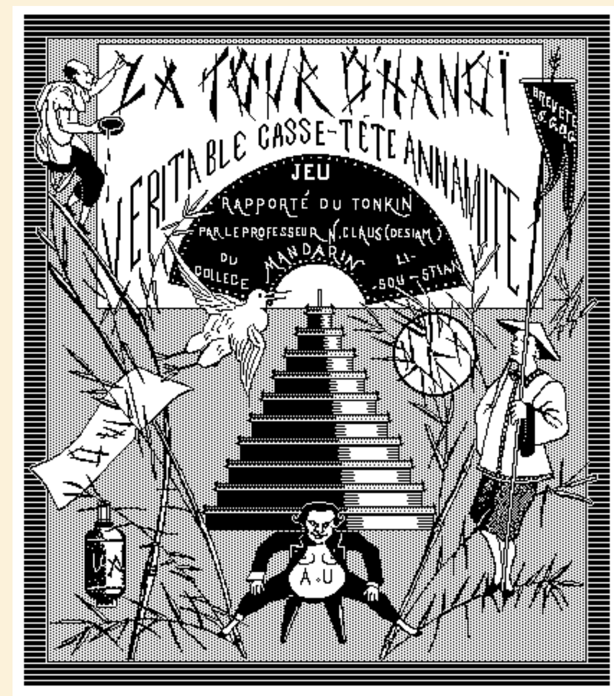
LinearFibonacci(k-1): $F_{k-1}$, $F_{k-2}$

- Runs in ***O*(*k*)** time.

# Binary Recursion

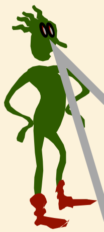
- Second Example: **The Tower of Hanoi**





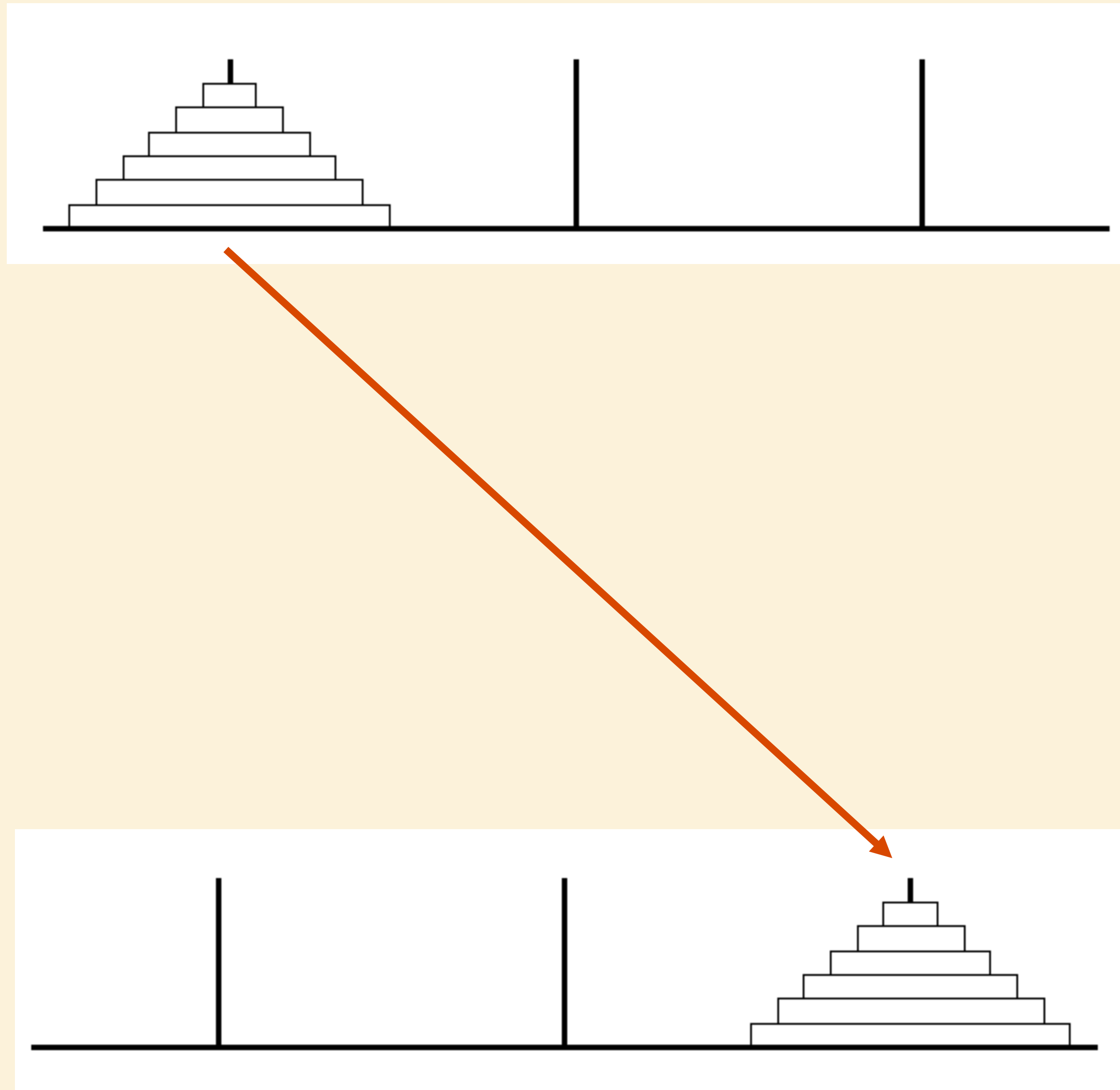
# Example

# Tower of Hanoi



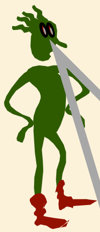
This job of mine is a bit daunting. Where do I start?

And I am lazy.

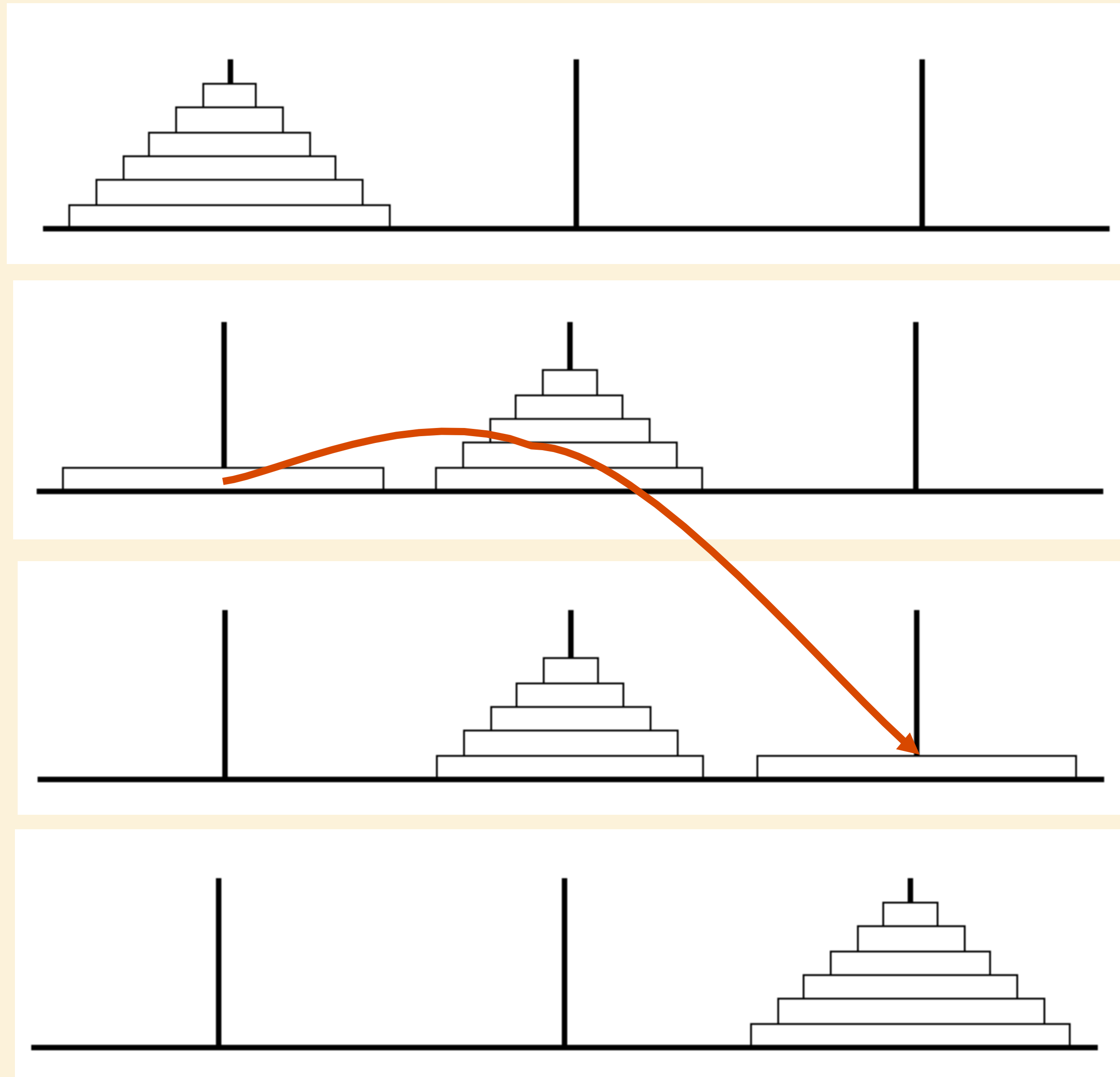


Example from J. Edmonds – Thanks Jeff!

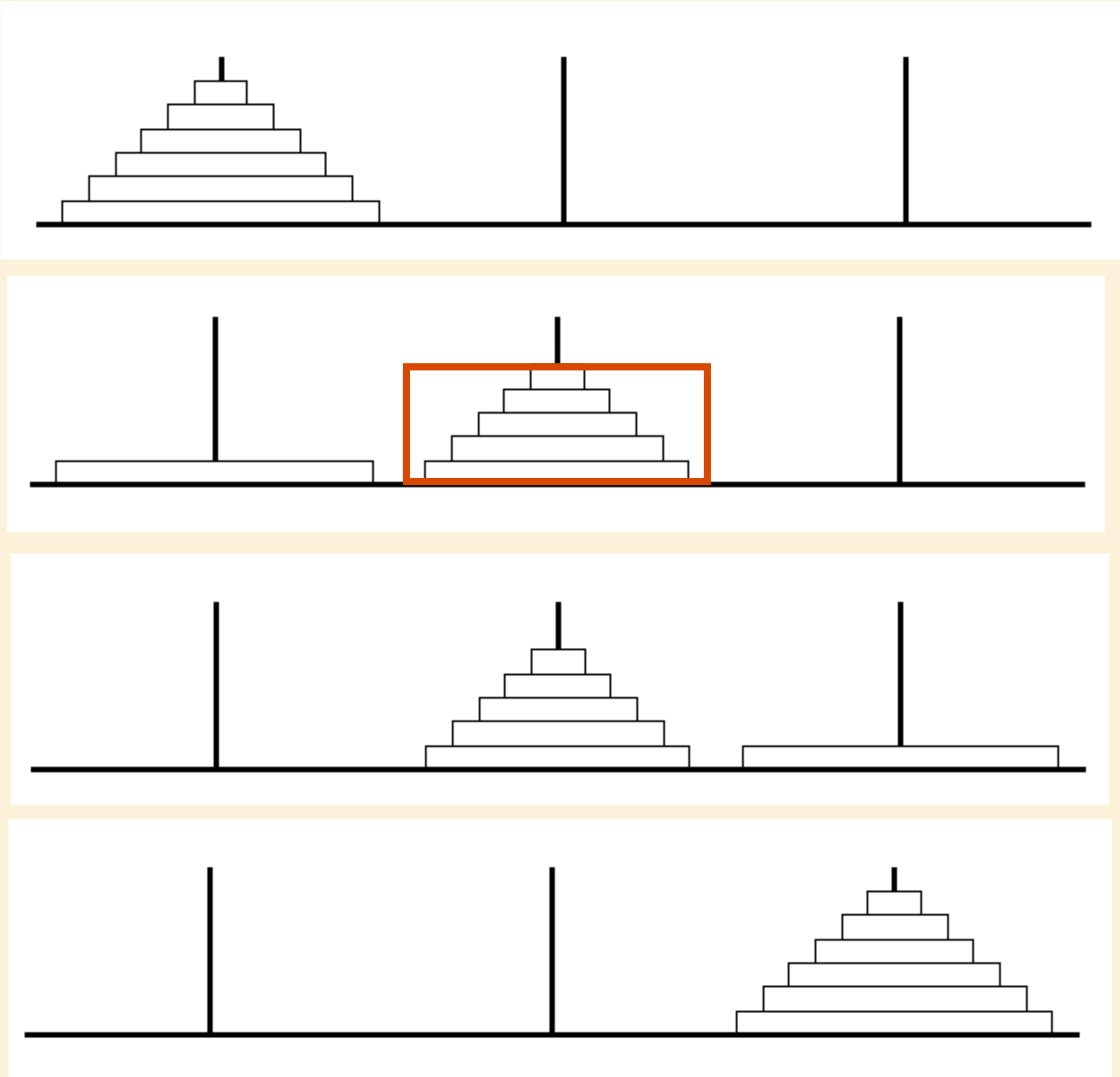
# Tower of Hanoi



At some point, the biggest disk moves.
I will do that job.

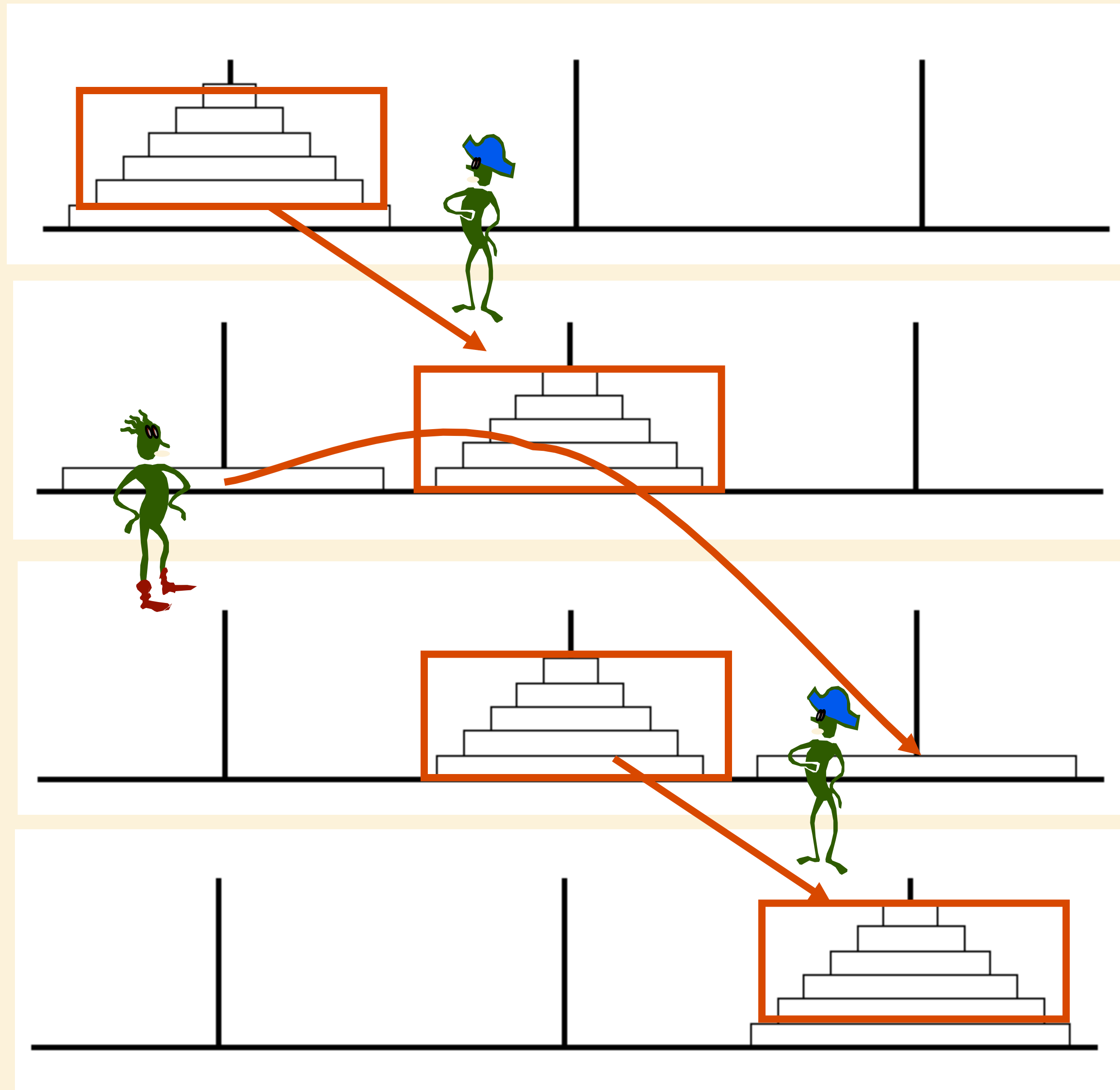


# Tower of Hanoi



To do this,
the other disks
must be in the
middle.

# Tower of Hanoi



How will these move?
I will get a friend to do it.
And another to move these.
I only move the big disk.

# Tower of Hanoi

**Code:**

**algorithm** $TowersOfHanoi(n, source, destination, spare)$

$\langle \boldsymbol{pre-cond} \rangle$: The $n$ smallest disks are on $pole_{source}$.

$\langle \boldsymbol{post-cond} \rangle$: They are moved to $pole_{destination}$.

```
begin
    if(n = 1)
        Move the single disk from pole_source to pole_destination.
    else
        TowersOfHanoi(n − 1, source, spare, destination)
        Move the n^th disk from pole_source to pole_destination.
        TowersOfHanoi(n − 1, spare, destination, source)
    end if
end algorithm
```

**2 recursive calls!**

# Tower of Hanoi

**Code:**

**algorithm** $TowersOfHanoi(n, source, destination, spare)$

$\langle \textbf{\textit{pre}} - \textbf{\textit{cond}} \rangle$: The $n$ smallest disks are on $pole_{source}$.

$\langle \textbf{\textit{post}} - \textbf{\textit{cond}} \rangle$: They are moved to $pole_{destination}$.

begin
  if($n = 1$)
    Move the single disk from $pole_{source}$ to $pole_{destination}$.
  else
    $TowersOfHanoi(n-1, source, spare, destination)$
    Move the $n^{th}$ disk from $pole_{source}$ to $pole_{destination}$.
    $TowersOfHanoi(n-1, spare, destination, source)$
  end if
end algorithm

Time:

$T(1) = 1,$

$T(n) = 1 + 2T(n-1) \approx 2T(n-1)$

$\approx 2(2T(n-2)) \approx 4T(n-2)$

$\approx 4(2T(n-3)) \approx 8T(n-3)$

$\approx 2^i\, T(n-i)$

$\approx 2^n$

Exponential again!!

# Binary Recursion: Summary

- Binary recursion spawns an exponential number of recursive calls.
- If the inputs are only declining **arithmetically** (e.g., n-1, n-2,…) the result will be an exponential running time.
- In order to use binary recursion, the input must be declining **geometrically** (e.g., n/2, n/4, …).
- We will see efficient examples of binary recursion with geometricaly declining inputs when we discuss **heaps** and **sorting**.

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# The Overhead Costs of Recursion

- Many problems are naturally defined recursively.
- This can lead to simple, elegant code.
- However, recursive solutions entail a **cost in time and memory**: each recursive call requires that the current process state (variables, program counter) be **pushed** onto the system stack, and **popped** once the recursion unwinds.
- This typically affects the running time **constants**, but **not** the **asymptotic time complexity** (e.g., $O(n)$, $O(n^2)$ etc.)
- Thus **recursive solutions may still be preferred** unless there are very strict time/memory constraints.

# The “Curse” in Recursion: Errors to Avoid

```java
// recursive factorial function
public static int  recursiveFactorial(int n) {
   return  n  *  recursiveFactorial(n- 1);
}
```

- **There must be a base condition: the recursion must ground out!**

# The "Curse" in Recursion: Errors to Avoid

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  recursiveFactorial(n); // base case
    else return  n  *  recursiveFactorial(n- 1); // recursive case
}
```

- **The base condition must not involve more recursion!**

# The "Curse" in Recursion: Errors to Avoid

```
// recursive factorial function
public static int  recursiveFactorial(int n) {
    if  (n  ==  0)  return  1;      // base case
    else return  (n – 1)  *  recursiveFactorial(n);          // recursive case
}
```

- The input **must be converging** toward the base condition!

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  - Identify the base case for an inductive solution
  - Design and analyze linear and binary recursion algorithms
  - Identify the overhead costs of recursion
  - Avoid errors commonly made in writing recursive algorithms